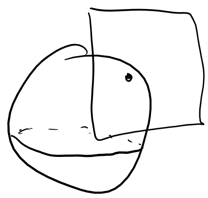
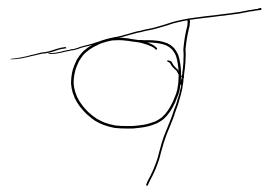
Introduction

Dienstag, 1. Dezember 2020 00:30

Characteristic classes

Q: When $TM \cong M \times \mathbb{R}^n$?

$TS^2 =$  $\not\cong S^2 \times \mathbb{R}^2$

 $= TS^1 \cong S^1 \times \mathbb{R}$

Goal: Develop algebraic invariants of Vector bundles (VB) called „Characteristic classes"

A characteristic of an n-dim real VB ξ is an element $c(\xi) \in H^q(B; G)$

Base $\nearrow$ $\quad \nwarrow$ $\mathbb{Z}_2$ or $\mathbb{Z}$

Def: A n-dim real Vector bundle (VB) is a fiber bundle in which each fiber has n-dim Vector space structure (fiber $\cong \mathbb{R}^n$) depending continuously on the basepoint.

Recall: A fiber bundle consists of top. spaces E (total space) B (base space), F (fiber) and a surj map $p: E \to B$ satisfying:

$\forall x \in B$, $\exists U \subseteq B$ open nbhd of x s.t. $\exists h$

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h", "\cong"'] \arrow[dr, "p"'] & \circlearrowleft & U \times F \arrow[dl, "pr"] \\ & U & \end{tikzcd}$$

Ex: (0) $B \times \mathbb{R}^n \xrightarrow[pr]{} B$ trivial VB

(1) tangent bundle of a smooth mfd M, TM

Def: $TM = \bigsqcup_{p \in M} T_p M$

(2) "Möbius VB"

(3) tautological bundle over $\mathbb{RP}^n$:

total space = pairs (ℓ, x) ℓ is a line in $\mathbb{R}^{n+1}$ through 0, $x \in \ell$

base $= \mathbb{R}P^n$

$p : (\ell, x) \longmapsto \ell$

fiber $= \mathbb{R}$

<u>Def</u>: Given a fiber bundle $\xi = (E, B, F, p)$ and a map $f : \tilde{B} \to B$ we define the induced bundle $f^* \xi$:

total space : $\{ (b', e) \in \tilde{B} \times E \mid f(b') = p(e) \}$

Base : $\tilde{B}$

p = projection

fiber $= F$

$$\begin{tikzcd} E' \arrow[r, "pr"] \arrow[d, "pr"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d] \\ \tilde{B} \arrow[r, "f"'] & B \end{tikzcd}$$

<u>Def</u>: Given 2 VB ξ_1, ξ_2 over the same base B. then

direct sum $:= \xi_1 \oplus \xi_2 := \Delta^* (\xi_1 \times \xi_2)$ $\Delta : B \to B \times B$, $x \mapsto (x, x)$

" $\xi_1 \oplus \xi_2$ = VB over B with fiber $F_1 \oplus F_2$ "

<u>Def</u>: A bundle map from ξ_1 to ξ_2 is a pair of maps F, f s.t.

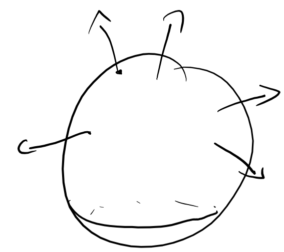
$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

$\underbrace{\qquad}_{\xi_1}$ $\underbrace{\qquad}_{\xi_2}$

If f and F are $\cong$, then $\xi_1 \cong \xi_2$

Def: Let M be smooth n-mfd embedded via i into a smooth mfd N. We define the normal bundle of M as:

$$NM := (TN)_{|M} / TM$$



$$B \times \mathbb{R} \longrightarrow B = \mathbb{1}$$

Def: We say that 2 VB ξ_1, ξ_2 over the same base are stably equivalent $\xi_1 \sim \xi_2$ if:

$$\xi_1 \oplus \mathbb{1} \oplus \dots \oplus \mathbb{1} \cong \xi_2$$

ξ_1 stably trivial $:\Longleftrightarrow$ ξ_1 is stably equivalent to trivial bundle

Remark: the normal bundle depends on the embedding only up to stable equivalence.

Ex: S^2, Σ_g are stably trivial

Def: Let $n, q \geq 0$, then a q-dim characteristic class of an n-dim | real / complex / oriented VB is a map which assigns to each n-dim real VB a class $c(\xi) \in H^q(\underset{\text{base}}{B}; \underset{\mathbb{Z} \text{ or } \mathbb{Z}_2}{G})$

s.t. $\forall f : \tilde{B} \to B$, $f^* c(\xi) = c(f^* \xi)$

$p \circ s = id_B$

Recall: Given a fiber bundle $\xi = (E, B, F, p)$, B CW-base and $\pi_0(F) = \dots = \pi_{n-1}(F) = 0$, $\pi_n(F) \neq 0$. then $\exists\, s : B^n \to E$

take $(n+1)$-cell. Get a map $\partial e = S^n \xrightarrow{h} B^n \xrightarrow{s} U \cong D^{n+1} \times F \simeq F$, $U \subseteq E$

$\Rightarrow$ Get an element in $\pi_n(F)$.

Do this $\forall$ cells, and get $c(\xi) := [C_{n+1}(B) \to \pi_n(F)] \in H^{n+1}(B; \pi_n(F))$

For VB, $F \cong \mathbb{R}^n \simeq pt \Rightarrow c(\xi) = 0$

Plan: take VB $\xi \rightsquigarrow \xi_k \rightsquigarrow \xi_k^0$ ← fiber $(n-k)$ connected

$0 \leq k \leq n$ $\quad$ $0 \leq k \leq n$

Stiefel Whitney classes $\to w_j(\xi) = c(\xi^0_{n-j+1}) \bmod 2 \in H^j(B; \mathbb{Z}_2)$

Construction of associated fibrations

Let $\xi = (E, B, \mathbb{R}^n, p)$ be a VB and let $1 \leq k \leq n$

$E_k = \{ (x_1, \dots, x_k) \in \underbrace{E \times \dots \times E}_{k} \mid p(x_1) = \dots = p(x_k) \quad x_1, \dots, x_k \text{ lin indep} \}$

there arises a fiber bundle $\xi_k = (E_k, B, \mathbb{R}_k, p_k)$

R_k is the space of all k-frames in $\mathbb{R}^n$

for $k=1$: $R_1 = \mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$

Lemma: If a VB ξ has a CW-base, then one can introduce a euclidean structure in each fiber depending continuously on the basepoint.

Using this Lemma, we introduce a euclidean structure in each fiber of ξ_k. Define the following fiber bundle

ξ_k^0
- total space = {all orthonormal k-frames in all fibers of ξ_k}
- Base = B
- fiber = {all orthonormal k-frames in $\mathbb{R}^n$} $=: V(n,k)$

$$V(n,1) = S^{n-1}$$

Case $k=1$: Assume ξ oriented. Then ξ_1^0 is also oriented.
fiber $= S^{n-1}$. We define the euler class:

$$e(\xi) := c(\xi_1^0) \in H^n(B;\mathbb{Z})$$

Lemma: (i) $\pi_i(V(n,k)) = 0 \quad \forall i < n-k$

(ii) $\pi_{n-k}(V(n,k)) \cong \begin{cases} \mathbb{Z}, & k=1 \text{ or } n-k \text{ even} \\ \mathbb{Z}_2, & \text{else} \end{cases}$

Def: We define $w_j(\xi) = c(\xi^0_{n-j+1}) \bmod 2 \in H^j(B;\mathbb{Z}_2)$

$w_0(\xi) = 1 \qquad$ for $j > \dim \xi \qquad w_j(\xi) := 0$

$$w(\xi) = w_0(\xi) + \dots + w_n(\xi) \in H^*(B; \mathbb{Z}_2)$$

<u>Proof of Lemma</u>: take a VB $\xi = (E, B, \mathbb{R}^n, p)$. Define

$\tilde{\xi}$ ← total space = {all euclidean structures in all fibers}
← base = B
← fiber = {all euclidean structures in a VS} ← contractible since convex

obstruction theory ⟹ $\exists s : B \to \tilde{\xi}$, thus we introduced a euclidean structure in each fiber

<u>Lemma</u>: Consider the fibration

$$V(n,r) \longrightarrow S^{n-1} \qquad \text{fiber } V(n-1, r-1)$$
$$(x_1, \dots, x_r) \longmapsto x_r$$

$$\dots \to \pi_i(S^{n-1}) \to \pi_i(V(n-1, r-1)) \to \pi_i(V(n,r)) \to \pi_{i-1}(S^{n-1})$$

$$c_j(\xi) := c(\xi^0_{n-j+1}) \in H^{2j}(B; \underset{\sim}{\mathbb{Z}})$$

$$w_j(\mathbb{R}\xi) = c_j(\xi) \bmod 2$$

What do they measure?

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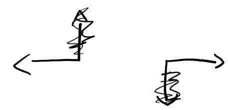
Prop: $w_1(\xi) = 0 \iff \xi$ orientable



Proof: $w_1(\xi) = c(\xi_n^1) \mod 2$

total space = {all orthonormal bases in all fibers}

ξ_n^1 ← fiber $= V(n,n) = O(n)$

← base B

ξ orientable $\iff \exists$ section $s : B^1 \to \xi_n^0$

$\iff w_1(\xi) = c(\xi_n^0) \mod 2 = 0 \in H^1(B;\mathbb{Z}_2)$

Remark: w_1 measures triviality of ξ over B^1

w_j measures how "twisted or non-trivial" the VB ξ is over the j-th skeleton.

Remark: $w_2(\xi) = 0 \iff \xi$ admits a Spin structure

What does $e(\xi) \in H^n(B;\mathbb{Z})$ measure?

Prop: $e(\xi) = 0 \iff \xi$ admits a non vanishing vector field

Proof: If $e(\xi) = 0 \overset{\text{no obstructions}}{\Rightarrow} \exists\, s : B \to \xi_1^0$ s is a section

$\parallel$

$c(\xi_1^0)$ $\Rightarrow \exists$ non vanishing vector field

Theorem: the Stiefel-Whitney classes have the following properties:

(i) For ζ (tautological bundle over $\mathbb{R}P^n$), $(n \geq 2)$

$$w_1(\zeta) \neq 0 \in H^1(\mathbb{R}P^n; \mathbb{Z}_2) \quad \text{and} \quad w_j(\zeta) = 0 \quad \forall j > 1$$

(ii) Let ξ, η be 2 VB over same base. Then

$$w_i(\xi \oplus \eta) = \sum_{p+q=i} w_p(\xi) \cup w_q(\eta)$$

$$w(\xi \oplus \eta) = w(\xi) \cup w(\eta)$$

Remark: (i) + (ii) + $\left(f^* w_j(\xi) = w_j(f^* \xi)\right)$ are often taken as axioms of the S-W-classes

See: Milnor - Stasheff

Proof: See Homotopical topology Fomenko

Prop: $w(T\mathbb{R}P^n) =: w(\mathbb{R}P^n) = (1+x)^{n+1}$

x is generator of $H^1(\mathbb{R}P^n; \mathbb{Z}_2)$

$$w_j(T\mathbb{R}P^n) = \binom{n+1}{j} x^j$$

$H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x] / (x^{n+1})$

$\deg(x) = 1$

Proof: $T\mathbb{R}P^n \oplus \underline{1} \cong \underbrace{\zeta \oplus \ldots \oplus \zeta}_{n+1} =: (n+1)\zeta$

$w(\zeta) = 1 + x$

$$\Rightarrow \quad w(T\mathbb{R}P^n) \overset{(ii)}{=} w(T\mathbb{R}P^n \oplus \underline{1}) = w((n+1)\zeta) \overset{(ii)}{=} (1+x)^{n+1}$$

$w(S^2) = w(\varepsilon_g) = 1$

$e(S^2) = 2[S^2] \in H^2(S^2; \mathbb{Z})$

$\langle e(M), [M] \rangle = \chi(M)$

Sources

Freitag, 27. November 2020 11:28

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-A. Hatcher - "Vector Bundles & K-Theory" https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf

-J.W. Milnor and J.D. Stasheff - "Characteristic Classes", Princeton University Press, 1974